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A UNIFYING APPROACH TO MULTIPARAMETER QUANTUM GROUPS

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1 – PRELIMINARIES (what's known)

— UNIPARAMETER QUANTUM GROUPS —

Our "quantum groups" are **QUEA**'s over some Lie algebras

We look at **semisimple Lie algebras**, **Kac-Moody algebras** and their kin — therefore we **FIX** the following

Cartan data

- $A := \left(a_{i,j}\right)_{i,j \in I} =$ a generalized symmetrizable Cartan matrix, $n := |I|$
- $D := diag\left(d_i\right)_{i \in I}$ diagonal matrix with "minimal" integral entries such that $D\,A$ is symmetric
- $\mathfrak{h} :=$ **"Cartan subalgebra"** attached with A, $t := rk(\mathfrak{h})$
- simple **roots** $\alpha_i \in \mathfrak{h}^*$ $(\, i \in I\,)$ & simple **coroots** $H_i \in \mathfrak{h}$ $(\, i \in I\,)$
- $\mathfrak{g} :=$ the **Kac-Moody algebra** associated with A and $\mathfrak{h}$

Drinfeld's (formal) QUEA

Def.: $U_\hbar(\mathfrak{g}) :=$ $\hbar$–complete Hopf algebra over $\Bbbk[[\hbar]]$ with

GENERATORS: $H\ (\in \mathfrak{h})\,,\ \ E_i\ (\, i \in I\,)\,,\ \ F_i\ (\, i \in I\,)$

RELATIONS: $\forall\ H, H', H'' \in \mathfrak{h}\,,\ i, j \in I\,,\ i \neq j$

$$H'\,H'' \;=\; H''\,H' \;\;, \quad E_i\,F_j \,-\, F_j\,E_i \;=\; \delta_{i,j}\,\frac{e^{+\hbar d_i H_i} - e^{-\hbar d_i H_i}}{e^{+\hbar d_i} - e^{-\hbar d_i}}$$

$$H\,E_j \,-\, E_j\,H \;=\; +\alpha_j(H)\,E_j \;\;, \quad H\,F_j \,-\, F_j\,H \;=\; -\alpha_j(H)\,F_j$$

$$\sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \begin{bmatrix} 1-a_{ij} \\ \ell \end{bmatrix}_{e^{+\hbar d_i}} X_i^{1-a_{ij}-\ell}\,X_j\,X_i^\ell \;=\; 0 \qquad \forall\ X \in \{E\,,F\}$$

HOPF STRUCTURE $(\,\forall\ H \in \mathfrak{h}\,,\ i \in I\,)$: $\quad \Delta(H) = H \otimes 1 + 1 \otimes H$

$$\Delta(E_i) = E_i \otimes 1 + e^{+\hbar d_i H_i} \otimes E_i \;\;, \qquad \Delta(F_i) = F_i \otimes e^{-\hbar d_i H_i} + 1 \otimes F_i$$

REMARKS: *(a)* $\exists$ ***"polynomial" version*** of $U_\hbar(\mathfrak{g})$, by Jimbo & Lusztig

(b) $\exists$ ***"quantum double version"*** of these QUEA's, both in formal and in polynomial formulation — roughly, you "duplicate" $\mathfrak{h}$

— FROM “UNI-” TO “MULTI-” —

Multiparameter QUEA — both “formal” and “polynomial” — were introduced by adding new “discrete” parameters to a 1-parameter QUEA.

***Formal* (Reshetikhin):** For any $\Psi := \left(\psi_{gk}\right)_{g,k=1,\dots,t} \in \mathfrak{so}_t\big(\Bbbk[[\hbar]]\big)$, $\mathfrak{g}$ of *finite type*, there is a **(formal) multiparameter QUEA**, say $U_\hbar^\Psi(\mathfrak{g})$, s.t.

(a) as an *algebra*, $U_\hbar^\Psi(\mathfrak{g})$ is the same as Drinfeld's $U_\hbar(\mathfrak{g})$

(b) $U_\hbar^\Psi(\mathfrak{g})$ has a “deformed” coproduct depending on the ψ_{gk}'s

***Polynomial* (Andruskiewitsch-Schneider & Al.):** For every matrix $\mathbf{q} := \left(q_{ij}\right)_{i,j\in I} \in M_n(\mathbb{K})$ such that $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$, there exists a **(polynomial) multiparameter QUEA**, say $\mathbf{U}_\mathbf{q}(\mathfrak{g})$, s.t.:

(a) as a *coalgebra*, $\mathbf{U}_\mathbf{q}(\mathfrak{g})$ is the same as the “quantum double version” of Jimbo-Lusztig's (polynomial) QUEA, denoted $\mathbf{U}_{\check{\mathbf{q}}}(\mathfrak{g})$

(b) $\mathbf{U}_\mathbf{q}(\mathfrak{g})$ has a “deformed” product depending on the q_{ij}'s

— DEFORMATION TECHNIQUES —

Definition: for every Hopf algebra H, we call:

(T) ***twist*** of H any $\mathcal{F} \in H \otimes H$ such that:

(T.1) $\mathcal{F}$ is invertible – *(T.2)* $(\epsilon \otimes id)(\mathcal{F}) = 1 = (id \otimes \epsilon)(\mathcal{F})$

(T.3) $(\mathcal{F} \otimes 1) \cdot (\Delta \otimes id)(\mathcal{F}) = (1 \otimes \mathcal{F}) \cdot (id \otimes \Delta)(\mathcal{F})$

(C) ***2-cocycle*** of H any $\sigma \in \big(H \otimes H\big)^*$ such that ($\forall\, a, b, c \in H$):

(C.1) σ is (convolution-)invertible – *(C.2)* $\sigma(a\,,1) = \epsilon(a) = \sigma(1\,,a)$

(C.3) $\sigma\big(b_{(1)}, c_{(1)}\big) \cdot \sigma\big(a\,, b_{(2)}\, c_{(2)}\big) = \sigma\big(a_{(1)}, b_{(1)}\big) \cdot \sigma\big(a_{(2)}\, b_{(2)}, c\big)$

Remarks: these notions are dual to each other...

FACT: *(deformations by twist / 2-cocycle)* Let H, $\mathcal{F}$, σ be as above:

(def.T–$\mathcal{F}$) the *algebra* H turns into a new Hopf algebra $H^{\mathcal{F}}$ with *new coproduct* $\Delta^{\mathcal{F}} := \mathcal{F} \cdot \Delta(-) \cdot \mathcal{F}^{-1}$

(def.C–σ) the *coalgebra* H turns into a new Hopf algebra H_σ with *new product* $m_\sigma := \sigma * m * \sigma^{-1}$

This gives a link between *multiparameter* QUEA's and *uniparameter* ones:

FACT: (*formal case*) for every $\Psi := \big(\psi_{ij}\big)_{i,j=1,\dots,t} \in \mathfrak{so}_t\big(\Bbbk[[\hbar]]\big)$, there exists a suitable twist $\mathcal{F}_\Psi$ of $U_\hbar(\mathfrak{g})$ such that $U_\hbar^\Psi(\mathfrak{g}) = \big(U_\hbar(\mathfrak{g})\big)^{\mathcal{F}_\Psi}$

(*polynomial case*) for every $\mathbf{q} := \big(q_{ij}\big)_{i,j\in I} \in M_n(\mathbb{K})$ such that $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$, there exists a suitable 2-cocycle $\sigma_{\mathbf{q}}$ of $\mathbf{U}_{\check{\mathbf{q}}}(\mathfrak{g})$ — where $\check{\mathbf{q}}$ is the "standard" multiparameter — such that $\mathbf{U}_{\mathbf{q}}(\mathfrak{g}) = \big(\mathbf{U}_{\check{\mathbf{q}}}(\mathfrak{g})\big)_{\sigma_{\mathbf{q}}}$

In a nutshell: *Any **multiparameter** QUEA (in the sense of Reshetikhin, resp. of Andruskiewitsch-Schneider) is a **deformation** of a **uniparameter** QUEA by twist, resp. by 2-cocycle*,

in short

***multiparameter** QUEA = **deformation of uniparameter** QUEA*

Remark: (formal/polynomial) multiparameter QUEA's can be realized as ***quantum/Drinfeld double*** of suitable "Borel" quantum (sub)groups.

2 – A UNIFYING APPROACH (what's new!)

Main Goal: find a notion of MpQUEA encompassing $U_\hbar^\Psi(\mathfrak{g})$ and $\mathbf{U_q}(\mathfrak{g})$

Results: *(1)* we do find such a good notion of MpQUEA $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$

(2) the family of all $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$'s is *stable under ("nice") deformations*

(3) specialization yields lots of *multiparameter Lie bialgebras*

Definition: Fix $P = \left(p_{ij}\right)_{i,j \in I} \in M_n(\Bbbk[[\hbar]])$ s.t. $P + P^t = 2\,D\,A$.
We define ***realization*** of P any triple $\mathcal{R} := \left(\mathfrak{h}\,,\,\Pi\,,\,\Pi^\vee\right)$ such that

— $\mathfrak{h}$ is a free module of finite rank over $\Bbbk[[\hbar]]$

— $\Pi := \{\alpha_i\}_{i \in I} \subseteq \mathfrak{h}^*$ (the set of simple *"roots"*)

— $\Pi^\vee := \{T_i^+, T_i^-\}_{i \in I} \subseteq \mathfrak{h}$ (the set of simple *"coroots"*)

— $\alpha_j(T_i^+) = p_{ij}$ & $\alpha_j(T_i^-) = p_{ji}$ for all $i, j \in I$

— *(...some extra technicalities...)*

N.B.: realizations of P naturally form a category

DEFINITION 1 / THEOREM 1: *(cf. [GaGa2], 2022)*

For $P = \big(p_{ij}\big)_{i,j \in I}$ and a realization $\mathcal{R} := \big(\mathfrak{h}\,,\,\Pi\,,\,\Pi^\vee\big)$ as above, we set

$U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) :=$ $\hbar$–adically complete unital associative $\Bbbk[[\hbar]]$–algebra with

GENERATORS: $\quad T\,(\in \mathfrak{h})\,,\;\; E_i\,(\,i \in I\,)\,,\;\; F_i\,(\,i \in I\,)$

RELATIONS $(\forall\; T, T', T'' \in \mathfrak{h}\,,\; i,j \in I\,,\; i \neq j\,)$:

$$T'\,T'' \;=\; T''\,T' \;\;, \quad E_i\,F_j \,-\, F_j\,E_i \;=\; \delta_{i,j}\,\frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{e^{+\hbar d_i} - e^{-\hbar d_i}}$$

$$T\,E_j \,-\, E_j\,T \;=\; +\alpha_j(T)\,E_j \;\;, \quad T\,F_j \,-\, F_j\,T \;=\; -\alpha_j(T)\,F_j$$

$$\sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \begin{bmatrix} 1-a_{ij} \\ \ell \end{bmatrix}_{e^{+\hbar d_i}} e^{+\hbar \ell (p_{ij}-p_{ji})/2}\, X_i^{1-a_{ij}-\ell}\, X_j\, X_i^\ell \;=\; 0 \;\;, \quad X \in \{E\,,F\}$$

HOPF STRUCTURE $(\forall\; T \in \mathfrak{h}\,,\; i \in I\,)$: $\quad \Delta(T) = T \otimes 1 + 1 \otimes T$

$$\Delta(E_i) = E_i \otimes 1 + e^{+\hbar T_i^+} \otimes E_i \;\;, \qquad \Delta(F_i) = F_i \otimes e^{-\hbar T_i^-} + 1 \otimes F_i$$

N.B.: I wrote "Theorem" because we must prove that the given coproduct (etc.) is well defined indeed (plus details)!

What about **PROOF(S)**???

We can provide **four** proofs, independent of each other.

1st proof: adapts the usual proofs for Drinfeld's $U_\hbar(\mathfrak{g})$

2nd proof: reduces to $\mathcal{R}$ of special form and then relies on the existence of Hopf structure for A-S's (polynomial) MpQUEA $\mathbf{U_q}(\mathfrak{g})$

3rd proof: provides an alternative construction of $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})$ as a *Drinfeld double* of suitable (formal) multiparameter quantum Borel (sub)algebras, endowed with a suitable Hopf structure

4th proof: is deduced (by "reverse engineering") from the stability under deformations of our whole family of MpQUEA's

3 – STABILITY by DEFORMATIONS

Definition: *(T)* Fix a basis $\{H_g\}_{g,k=1,\dots,t}$ of $\mathfrak{h}$, $t := rk(\mathfrak{h})$; for every $\Phi = (\phi_{gk})_{g,k=1,\dots,t} \in \mathfrak{so}_t(\Bbbk[[\hbar]])$, we call *"toral" twist* of $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})$ the element $\mathcal{F}_\Phi := \exp\left(\hbar \sum_{g,k=1}^{t} \phi_{gk} H_g \otimes H_k\right)$

(C) Fix $\chi \in (\mathfrak{h} \wedge \mathfrak{h})^*$ s.t. $\chi(T_i^+ + T_i^-, -) = 0 = \chi(-, T_i^+ + T_i^-)$: it extends trivially to a 2-cocycle of $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})$. Then $\sigma_\chi := \exp_*(\hbar^{-1}\chi)$ is a $\Bbbk((\hbar))$*–valued* 2-cocycle of $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})$, that we call *"toral" 2-cocycle.*

THEOREM 2: *(stability for toral deformations — cf. [GaGa2])*

There is a matrix P_Φ, resp. $P_{(\chi)}$, a realization $\mathcal{R}_\Phi = (\mathfrak{h}, \Pi_\Phi = \Pi, \Pi_\Phi^\vee)$, resp. $\mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi_{(\chi)}^\vee = \Pi^\vee)$, of it and an explicit isomorphism

$$\left(U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})\right)^{\mathcal{F}_\Phi} \cong U^{\mathcal{R}_\Phi}_{P_\Phi,\hbar}(\mathfrak{g}) \quad , \qquad \text{resp.} \qquad \left(U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})\right)_{\sigma_\chi} \cong U^{\mathcal{R}_{(\chi)}}_{P_{(\chi)},\hbar}(\mathfrak{g})$$

In particular, every deformation by toral twist, resp. by toral 2-cocycle, of a FoMpQUEA is again another FoMpQUEA.

— PROOF —

— *for (toral) 2-cocycles:* not surprising, just needs careful computations...

— *for (toral) twists:* it exploits a **key idea**, which goes as follows:

(1) for the algebra structure alone we have $\big(U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})\big)^{\mathcal{F}_\Phi} = U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})$, hence in particular $\big(U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})\big)^{\mathcal{F}_\Phi}$ has the same generators as $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})$

(2) the generators T, E_i and F_i of $\big(U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})\big)^{\mathcal{F}_\Phi}$ are primitive (the T's) or (h,k)-skew-primitive (the E_i's and F_i's) for the new coproduct $\Delta^{\mathcal{F}_\Phi}$

(3) computations along with *(2)* show that $\big(U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})\big)^{\mathcal{F}_\Phi}$ and $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})$ have similar coradical filtration and same associated graded Hopf algebra

(4) by *(1–3)* we can modify the (h,k)-skew-primitive generators E_i and F_i into new generators E_i^Φ and F_i^Φ that are (h',k')-skew-primitive with $h'=1$ or $k'=1$, as it is for the E_i's and the F_i's in any FoMpQUEA

(5) the new generators T, E_i^Φ and F_i^Φ obey the relations that rule $U^{\mathcal{R}_\Phi}_{P_\Phi,\hbar}(\mathfrak{g})$, with a simultaneous choice of suitable new "(simple) coroots"

So an isomorphism $\left(U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})\right)^{\mathcal{F}_\Phi} \xleftarrow{\;\cong\;} U^{\mathcal{R}_\Phi}_{P_\Phi,\hbar}(\mathfrak{g})$ is defined by mapping the generators of $U^{\mathcal{R}_\Phi}_{P_\Phi,\hbar}(\mathfrak{g})$ onto the *new* generators of $\left(U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})\right)^{\mathcal{F}_\Phi}$

In short, the isomorphism $\left(U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})\right)^{\mathcal{F}_\Phi} \cong U^{\mathcal{R}_\Phi}_{P_\Phi,\hbar}(\mathfrak{g})$ boils down to a

<u>change of presentation for $\left(U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})\right)^{\mathcal{F}_\Phi}$</u>

induced by a change of generators and a change of "(simple) coroots"

REMARKS

(1) Our FoMpQUEA $U^{\mathcal{R}}_{P,\hbar}(\mathfrak{g})$ is defined by letting
— the *algebra* structure depend on the parameters p_{ij}
— the coalgebra structure be kept fixed

Applying a toral *2-cocycle* deformation amounts to modifying the p_{ij}'s.
Instead, applying a toral *twist* deformation by $\mathcal{F}_\Phi$, we get
— the *algebra* structure (is the same, so) depends on the p_{ij}'s
— the *coalgebra* structure depends on the ϕ_{gk}'s
so the final object is described via a *double multiparameter* $(P \,|\, \Phi)$.

Nonetheless, Theorem 2 proves that, instead of $(P \,|\, \Phi)$, **a *"single"* (deformed) multiparameter P_Φ is enough**.

(2) The "standard" FoMpQUEA (with $P := DA$) is the double "lift" of Drinfeld's $U_\hbar(\mathfrak{g})$. *Under mild assumptions on $\mathcal{R}$*, Theorems 2 implies

— *every $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is a* ***2-cocycle deform.*** *of the "standard" FoMpQUEA*

$\Longrightarrow$ *$U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ admits a "fully polarized" presentation* with "discrete" parameters that rule the *algebra* structure, whereas the coalgebra structure is constant (*"à la Andruskiewitsch-Schneider"*),

— *every $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is a* ***twist deformation*** *of the "standard" FoMpQUEA*

$\Longrightarrow$ *$U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ admits a "fully polarized" presentation* with "discrete" parameters that rule the *coalgebra* structure, whereas the algebra structure is constant (*"à la Reshetikhin"*).

N.B.: we *chose* to define our notion of FoMpQUEA with a presentation of the *first* type, but the other option is available as well

4 – MULTIPARAMETER LIE BIALGEBRAS

Plan: we introduce Lie bialgebras with common "socle" the Manin double "lift" of a Kac-Moody algebra, with Lie coalgebra structure by Sklyanin-Drinfeld and Lie algebra structure depending on some parameters.

DEFINITION 2 / THEOREM 3: *(cf. [GaGa2], 2022)*

Fix $P = \big(p_{ij}\big)_{i,j \in I}$ and a realization $\mathcal{R} := \big(\mathfrak{h}\,,\Pi\,,\Pi^\vee\big)$ as before. We set

$\mathfrak{g}_P^{\mathcal{R}}$:= Lie algebra over $\Bbbk$ with generators $T \, (\in \mathfrak{h})$, $E_i \, (i \in I)$, $F_i \, (i \in I)$ and relations $\big(\forall\, T, T', T'' \in \mathfrak{h}\,,\ i,j,t \in I\,,\ i \neq t\big)$

$$[T', T''] = 0 \ , \quad [T, E_j] = +\alpha_j(T)\, E_j \ , \quad [T, F_j] = -\alpha_j(T)\, F_j$$

$$\big(ad(X_i)\big)^{1-a_{ij}}(X_j) = 0 \quad \big(X \in \{E, F\}\big) \ , \qquad [E_i, F_j] = \delta_{ij} \, \frac{T_i^+ + T_i^-}{2\, d_i}$$

Then *there exists a unique Lie bialgebra structure on* $\mathfrak{g}_P^{\mathcal{R}}$ *with Lie cobracket*

$$\delta(T) = 0 \ , \quad \delta(E_i) = 2\, T_i^+ \wedge E_i \ , \quad \delta(F_i) = 2\, T_i^- \wedge F_i \qquad (\forall\, T\,, i)$$

PROOF(S)??? We have **three** proofs, independent of each other!

1st proof: we provide an alternative construction of $\mathfrak{g}_P^{\mathcal{R}}$ itself (after reducing to special $\mathcal{R}$) as a *Manin's double* of multiparameter Borel (sub)algebras $\mathfrak{b}_{+,P}^{\mathcal{R}}$ and $\mathfrak{b}_{-,P}^{\mathcal{R}}$, endowed with a Lie bialgebra structure

2nd proof: another proof is deduced *a posteriori* — by "reverse engineering" — from the *stability under deformations* (see later!)

3rd proof: again *a posteriori*, another proof comes for free once we realize that $U(\mathfrak{g}_P^{\mathcal{R}})$ *is nothing but the semiclassical limit of* $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$

Long story short, the following holds (with ***Proof*** by direct inspection):

THEOREM 4: *(MpLbA's as semiclassical limits — cf. [GaGa2])*

The specialization of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ at $\hbar = 0$ is nothing but $U(\mathfrak{g}_P^{\mathcal{R}})$.
In other words, $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ is a quantization of $U(\mathfrak{g}_P^{\mathcal{R}})$.

N.B.: indeed, the story went the other way round: computing the semi-classical limit of $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})$ lead us to the description of the Lie bialgebra $\mathfrak{g}_P^{\mathcal{R}}$

— STABILITY by (toral) DEFORMATIONS —

Definition: for every Lie bialgebra $\mathfrak{g}$, we call:

(T) ***twist*** of $\mathfrak{g}$ any $c \in \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\mathrm{ad}_x\big((id \otimes \delta)(c) + \text{c.p.} + [\![c \,, c]\!]\big) = 0 \;, \quad \mathrm{ad}_x\big(c + c_{2,1}\big) = 0 \quad \forall\; x \in \mathfrak{g}$$

(C) ***2-cocycle*** of $\mathfrak{g}$ any $\gamma \in (\mathfrak{g} \otimes \mathfrak{g})^*$ such that

$$\mathrm{ad}_\psi\big(\partial_*(\gamma) + [\![\gamma \,, \gamma]\!]_*\big) = 0 \;, \qquad \mathrm{ad}_\psi\big(\gamma + \gamma_{2,1}\big) = 0 \qquad \forall\; \psi \in \mathfrak{g}^*$$

where $[\![r, s]\!] := [r_{1,2}, s_{1,3}] + [r_{1,2}, s_{2,3}] + [r_{1,3}, s_{2,3}]$ for any $r, s \in \mathfrak{g} \wedge \mathfrak{g}$

These gadgets are used to define *deformations*:

FACT: *(deformations by twist / 2-cocycle)* For every $\mathfrak{g}$, c and γ as above,

(def.T–c) the *Lie algebra* $\mathfrak{g}$ turns into a new Lie bialgebra $\mathfrak{g}^c$ with

$$\delta^{\,c} := \delta - \partial(c) \;, \quad \text{i.e. } \delta^{\,c}(x) := \delta(x) - \mathrm{ad}_x(c) \qquad \forall\; x \in \mathfrak{g}$$

(def.C–γ) the *Lie coalgebra* $\mathfrak{g}$ turns into a new Lie bialgebra $\mathfrak{g}_\gamma$ with

$$[x, y]_\gamma := [x, y] + \gamma\big(x_{[1]}, y\big)\, x_{[2]} - \gamma\big(y_{[1]}, x\big)\, y_{[2]} \qquad \forall\; x, y \in \mathfrak{g}$$

For MpLbA's, we consider a special type of "toral" twists & 2-cocycles:

Definition: *("toral" twists & 2-cocycles for MpLbA's)*

(T) For each $\Phi = \left(\phi_{gk}\right)_{g,k=1,\dots,t} \in \mathfrak{so}_t\big(\Bbbk[[\hbar]]\big)$, the element $c_\Phi := \sum_{g,k=1}^t \phi_{gk}\, H_g \otimes H_k$ is a *twist* of $\mathfrak{g}_P^{\mathcal{R}}$, that we call *"toral" twist*

(C) Any $\chi \in \left(\mathfrak{h} \wedge \mathfrak{h}\right)^*$ s.t. $\chi\big(T_i^+ + T_i^-, -\big) = 0 = \chi\big(-, T_i^+ + T_i^-\big)$ does extend trivially to a 2-cocycle γ_χ of $\mathfrak{g}_P^{\mathcal{R}}$, that we call *"toral" 2-cocycle*

Here is our **stability result**:

THEOREM 5: *(stability for toral deform.'s — cf. [GaGa2])*

There exist explicit isomorphisms $\left(\mathfrak{g}_P^{\mathcal{R}}\right)^{c_\Phi} \cong \mathfrak{g}_{P_\Phi}^{\mathcal{R}_\Phi}$ and $\left(\mathfrak{g}_P^{\mathcal{R}}\right)_{\gamma_\chi} \cong \mathfrak{g}_{P_{(\chi)}}^{\mathcal{R}_{(\chi)}}$

In particular, every deformation of a MpLbA by a (toral) twist or a (toral) 2-cocycle is again another MpLbA.

5 – SPECIALIZATION vs. DEFORMATION

The following diagram captures the overall picture

$$\begin{array}{ccccc} U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})^{\mathcal{F}_\Phi} & \overset{\textit{deformation}}{\longleftrightarrow} & U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g}) & \overset{\textit{deformation}}{\longleftrightarrow} & U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})_{\sigma_\chi} \\ \uparrow \textit{quantization} \uparrow \;\; \updownarrow \;\; \downarrow \textit{specialization} \downarrow & \circlearrowright & \updownarrow & \circlearrowleft & \updownarrow \;\; \uparrow \textit{quantization} \uparrow \;\; \downarrow \textit{specialization} \downarrow \\ U\big((\mathfrak{g}_P^{\mathcal{R}})^{c_\Phi}\big) & \underset{\textit{deformation}}{\longleftrightarrow} & U\big(\mathfrak{g}_P^{\mathcal{R}}\big) & \underset{\textit{deformation}}{\longleftrightarrow} & U\big((\mathfrak{g}_P^{\mathcal{R}})_{\gamma_\chi}\big) \end{array}$$

because $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})^{\mathcal{F}_\Phi} \cong U_{P_\Phi,\hbar}^{\mathcal{R}_\Phi}(\mathfrak{g})$, $U_{P,\hbar}^{\mathcal{R}}(\mathfrak{g})_{\sigma_\chi} \cong U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)}}(\mathfrak{g})$ — by Theorems 2 & 3 — and $(\mathfrak{g}_P^{\mathcal{R}})^{c_\Phi} \cong \mathfrak{g}_{P_\Phi}^{\mathcal{R}_\Phi}$, $(\mathfrak{g}_P^{\mathcal{R}})_{\gamma_\chi} \cong \mathfrak{g}_{P_{(\chi)}}^{\mathcal{R}_{(\chi)}}$ — by Theorems 6 & 7

...but *also* thanks to the following, general result:

THEOREM 6: *(cf. [GaGa2], 2022)*

For any QUEA $U_\hbar(\mathfrak{g})$, every twist / 2-cocycle of the Hopf algebra $U_\hbar(\mathfrak{g})$ induces by specialization a twist / 2-cocycle of the Lie bialgebra $\mathfrak{g}$. Then the process of specialization "commutes" with deformation (of either type)

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